AIAA JOURNAL
Vol. 33, No. 4, April 1995

Stability of Viscoelastic Plate in Supersonic Flow Under Random Loading

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The stability of a viscoelastic plate, subjected to a random load, in a supersonic flow is investigated. The stationary load is assumed in the form of Gaussian white noise. The relaxation kernel of the plate material is represented by a sum of exponents. The influence of such parameters as external damping, material viscosity, flow speed, and probabilistic characteristics of the load on the stability of the plate, in mean square form, is analyzed.

Nomenclature

D	= bending stiffness of the plate
k	= coefficient of the external and aerodynamic damping
l	= width of the plate
q	= uniform external load
$\boldsymbol{R}$	= relaxation operator
$R(t-\tau)$	= relaxation kernel
t	= time
u	= coefficient, depending on the flow speed
v	= speed of the flow
w	= transverse deflection of the plate
x	= coordinate
δ	= thickness of the plate
$æ_k^*$, L_k	= parameters of the kernel of the relaxation
ρ	= mass per unit of the plate volume

Introduction

THE stability of plates and shells in supersonic flow was considered in many works.[1–3] The analysis of the motion of structures under deterministic treatment of the problem is given in these works. The mathematical modeling is based on linear representation of elastic and aerodynamic forces. The aerodynamic load is obtained from a quasisteady first-order aerodynamic piston theory. Attention was focused on the study of the influence of the internal and external damping on the value of the critical load or flow speed. The effect of periodic in-plane forces has been studied for panels by Dżygadlo and Kaliski[4] and Dżygadlo,[5] who found the stability conditions for subsonic and supersonic flow.

Since the considered problem is a particular case of nonconservative problems, we cite the articles of Ziegler,[6] Bolotin and Zhinzher,[7] and Plaut,[8] which have principal significance for this direction of the stability investigation.

The number of works concerned with stability of plates subjected to stochastic loads is much smaller, though such problems are of significant theoretical and applied interest. Among those works, investigations by Plaut and Infante,[9] Kozin,[10] and Ahmadi[11] must be named. They considered elastic plates subjected to a load, which is a random stationary ergodic process. Authors obtained conditions of almost sure stability of plates on the basis of Lyapunov's second method.

The stochastic flutter of elastic panels represented by two modes interaction was examined by Ibrahim et al.[12] and Ibrahim and Orono.[13] The internal damping of the material was taken into account with the help of Voigt's model.[14] The random load was assumed in the form of Gaussian white noise. The response moment equations were generated by using the Fokker–Plank equation approach.

Received Oct. 14, 1993; revision received April 7, 1994; accepted for publication May 6, 1994.

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In Refs. 12, 13, and 15, stochastic nonlinear flutter of elastic panels subjected to random in-plane forces was considered.

The present article is devoted to the investigation of the stability of a long viscoelastic plate subjected to a random load in the form of stationary Gaussian process of the type white noise. The influence of viscoelastic properties of the plate material, the external and aerodynamic damping, the flow speed, and probabilistic characteristic of the load on the value of critical parameters, defined the stability in mean square of the plates, is discussed.

Equations of Motion of a Viscoelastic Plate in Supersonic Flow

Let us consider an infinitely long viscoelastic plate, freely supported at the long edges, in supersonic flow, moved with a constant speed v (Fig. 1). A uniform load $q(t)$ is applied to the mobile edges in the level of the middle flatness. Confining ourselves to the case of the cylindrical bending, we will assume that the transverse deflection of the plate w is the function of the coordinate x and time t, i.e., $w = w(x, t)$.

Using the piston theory, the equation of the plate motion, in the case of small displacements, is written in the following form:

$$D(1-\boldsymbol{R})\frac{\partial^4 w}{\partial x^4} + q\frac{\partial^2 w}{\partial x^2} + \rho\delta\frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} = 0 \tag{1}$$

where

$$\boldsymbol{R}\psi = \int_0^t R(t-\tau)\psi(\tau)\,\mathrm{d}\tau$$

$$0 \le \int_0^\infty R(\vartheta)\,\mathrm{d}\vartheta < 1$$

The solution of Eq. (1) must satisfy corresponding initial and boundary conditions too.

The deflection w is searched in the form of the expression

$$w(x,t) = \sum_{n=1}^{m} f_n(t)\sin\frac{n\pi}{l}x \tag{2}$$

which satisfies boundary conditions at $x = 0$ and $x = l$.

With help of the Bubnov–Galerkin method, we obtain from Eq. (1)

$$\frac{\mathrm{d}^2 f_n}{\mathrm{d}t^2} + 2\varepsilon^*\frac{\mathrm{d}f_n}{\mathrm{d}t} + \omega_n^2\left[(1-\boldsymbol{R}) - \frac{\alpha}{n^2}\right]f_n + \frac{1}{\rho\delta}\sum_{j=i}^{m} b_{nj}^* f_j = 0 \tag{3}$$

where

$$2\varepsilon^* = \frac{k}{\rho\delta}, \qquad \omega_n^2 = \frac{D}{\rho\delta}\left(\frac{n\pi}{l}\right)^4, \qquad \alpha = \frac{ql^2}{\pi^2 D}$$

$$b_{nj}^* = \frac{2\pi u j}{l^2}\int_0^l \sin\frac{n\pi}{l}x\cos\frac{j\pi}{l}x\,\mathrm{d}x$$

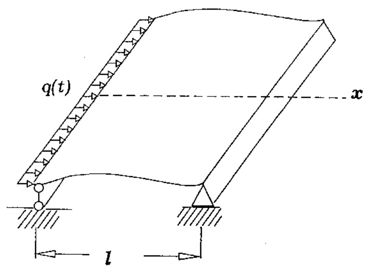

Fig. 1 Model of a plate in supersonic flow.

Let us assume that the kernel of the material relaxation $R(t-\tau)$ is represented by the sum of exponents

$$R(t-\tau)=\sum_{k=1}^{i}\text{æ}_k^* L_k \exp[-\text{æ}_k^*(t-\tau)] \tag{4}$$

Let us introduce new variables[16–18]

$$z_{kn}=\text{æ}_k^* L_k\int_0^t \exp[-\text{æ}_k^*(t-\tau)]f_n(\tau)\,\mathrm{d}\tau$$

Then

$$Rf_n=\sum_{k=1}^{i} z_{kn}$$

It is obvious that functions z_{kn} satisfy differential equations

$$\frac{\mathrm{d}z_{kn}}{\mathrm{d}t}=\text{æ}_k^* L_k f_n-\text{æ}_k^* z_{kn} \tag{5}$$

If we use the nondimensional time $t_1=\omega_1 t$, Eq. (3) is rewritten in this case in the following way:

$$f_n''+2\varepsilon f_n'+\mu_n\left[\left(1-\frac{\alpha}{n^2}\right)f_n-\sum_{k=1}^{i}z_{kn}\right]+\gamma\sum_{j=1}^{m}b_{nj}f_j=0 \tag{6}$$

Here

$$2\varepsilon=\frac{2\varepsilon^*}{\omega_1},\qquad \mu_n=\frac{\omega_n^2}{\omega_1^2}=n^4,\qquad \gamma=\frac{4ul^3}{\pi^4 D}$$

$$b_{nj}=\frac{j}{2}\int_0^{\pi}\sin nx\cos jx\,\mathrm{d}x$$

$$=\begin{cases}\dfrac{nj}{n^2-j^2}, & (n\pm j)\ \text{is an odd number}\\ 0, & (n\pm j)\ \text{is an even number}\end{cases}$$

The prime denotes differentiation with respect to time t_1.

Let us substitute Eqs. (5) and (6) with the system of differential equations of the first order:

$$\left.\begin{aligned} f_n'&=\psi_n\\ \psi_n'&=-2\varepsilon\psi_n-n^4\left[\left(1-\frac{\alpha}{n^2}\right)f_n-\sum_{k=1}^{i}z_{kn}\right]-\gamma\sum_{j=1}^{m}b_{nj}f_j\\ z_{kn}'&=\text{æ}_k L_k f_n-\text{æ}_k z_{kn}\end{aligned}\right\} \tag{7}$$

where $\text{æ}_k=\text{æ}_k^*/\omega_1$.

Let us use the vector form for the system (7)

$$x'=Ax \tag{8}$$

Here $x^T=(f_1\cdots f_m\,\psi_1\cdots\psi_m\,z_{11}\cdots z_{i1}\cdots z_{1m}\cdots z_{im})\equiv(x_1x_2\cdots x_{(2+i)m})$. The index "$T$" denotes transpose.

For an example at $m=2$ and $i=1$, vector x and matrix A have the form

$$x^T=(f_1 f_2\psi_1\psi_2 z_{11}z_{12})\equiv(x_1x_2\cdots x_6)$$

$$A=\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\-(1-\alpha)&\dfrac{2\gamma}{3}&-2\varepsilon&0&1&0\\-\dfrac{2\gamma}{3}&-16\left(1-\dfrac{\alpha}{4}\right)&0&-2\varepsilon&0&16\\ \text{æ}L&0&0&0&-\text{æ}&0\\0&\text{æ}L&0&0&0&-\text{æ}\end{bmatrix} \tag{9}$$

The equality $i=1$ denotes that the plate material is the standard viscoelastic one.

Further let us assume that the load q is a random stationary process in form of the white noise. Then

$$\alpha(t_1)=\alpha_0+\beta\xi$$

where $\alpha_0=\langle\alpha(t_1)\rangle\equiv\text{const}$, β is constant, and ξ is Gaussian white noise of the unit density, whose mathematical expectation is equal to zero.

Angle brackets denote the expected value (ensemble average).

Then Eq. (8) can be written in the form

$$x'=Cx+Bx\xi \tag{10}$$

The constant matrix C coincides with the matrix A at $\alpha=\alpha_0$. The matrix B has a simple structure. For example, at $m=2$ and $i=1$, it is written

$$B=\beta\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&4&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}[6pt] \tag{11}$$

Equations with Respect to Statistical Moments

From the system (10) on base Ito's formula,[19] systems of equations with respect to statistical moments of different order of functions $x_n(t)$ may be obtained simply, in particular systems, with respect to mean values $\langle x_n\rangle$ and to correlation moments $\langle x_n x_j\rangle, n, j=1,2,\ldots,(2+i)m$. It is important to notice that the supposition about the delta correlation of exterior load make it possible to obtain separate systems of the equations with respect to moment functions of each order.

The mean values $\langle x_n\rangle$ are found from the equation

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\langle x\rangle=C\langle x\rangle \tag{12}$$

where $\langle x\rangle$ is the vector, components of which are equal to $\langle x_n\rangle$. The system for moments of the second order is formed in the following way[19]:

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\langle x_n x_j\rangle=\sum_{s=1}^{p}(C_{ns}\langle x_j x_s\rangle+C_{js}\langle x_n x_s\rangle)$$
$$+\sum_{s=1}^{p}\sum_{r=1}^{p}b_{ns}b_{jr}\langle x_s x_r\rangle,\qquad p=(2+i)m \tag{13}$$

The system (13) may be written in the vector form

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\langle y\rangle=Dy \tag{14}$$

In principle, there are no difficulties to obtain systems of equations of statistical moments of the order higher than second. However, difficulties of the investigation of the stability of the solution of such systems may be so great that the solution of the problem is practically impossible.

Let us give the definition of the motion stability of the viscoelastic plate.

The motion of the plate is called k-stable[19] if for any $\varepsilon > 0$ can be found such $\Delta > 0$ that at $t \geq 0$ and $|x_j(0)| < \Delta$ inequality

$$m_k = |\underbrace{\langle x_j(t)x_p(t)\cdots\rangle}_{k\ \text{terms}}| < \varepsilon$$

is valid.

The motion of the plate is called asymptotically k-stable, if it is k-stable and, in addition, at small enough $|x_i(0)|$, the condition

$$\lim_{t\to\infty} m_k = 0$$

is satisfied.

At $k = 1$ we have the stability in mean and at $k = 2$ the stability in mean square.

Necessary and sufficient conditions of the asymptotical stability in mean and in mean square of the plate motion are satisfied, if real parts of roots of the characteristic equations

$$|C - \lambda I| = 0 \tag{15}$$

$$|D - \lambda I| = 0 \tag{16}$$

have negative sign.

Here I is the unit matrix.

The examination of these conditions may be made, for instance, with help of Routh–Hurvitz's criterion.

From the Eq. (15) we have the fact that conditions of the stability in mean of the motion of the plate, subjected to the random load, coincide with stability conditions of the same plate obtained from the deterministic problem, if the nondimensional value of the load is equal to α_0.

Example

Let us investigate the influence of different parameters on the plate stability, if the matrices C and B have the form of Eqs. (9) and (11). Since the stability of plates in supersonic flow in deterministical treatment of the problem was analyzed in enough detailed earlier, our attention will be concentrated on the stability in mean square.

The results of the numerical solution of the system of Eqs. (13) and (16) are presented in Figs. 2–7.

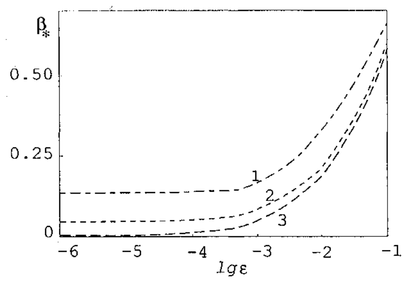

Fig. 2 Critical value β_* as a function of the parameter ε at α_0 = 0 and L = 0.5 (1 – $lg\,æ$ = –1, 2 – $lg\,æ$ = –2, and 3 – $lg\,æ$ = –6).

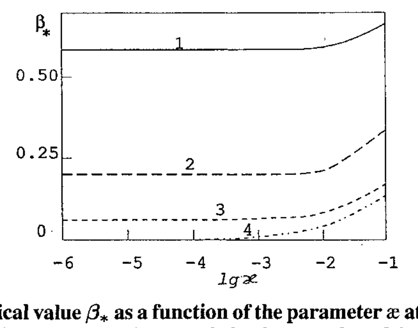

Fig. 3 Critical value β_* as a function of the parameter æ at α_0 = 0 and L = 0.5 (1 – $lg\,\varepsilon$ = –1, 2 – $lg\,\varepsilon$ = –2, 3 – $lg\,\varepsilon$ = –3, and 4 – $lg\,\varepsilon$ = –6).

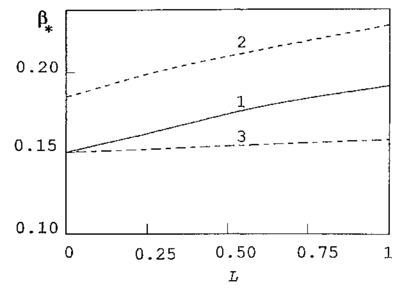

Fig. 4 Critical value β_* as a function of the parameter L (1 – α_0 = 0, ε = æ = 0.01; 2 – α_0 = 1, ε = æ = 0.01; and 3 – α_0 = 1, ε = 0.01, æ = 0.001).

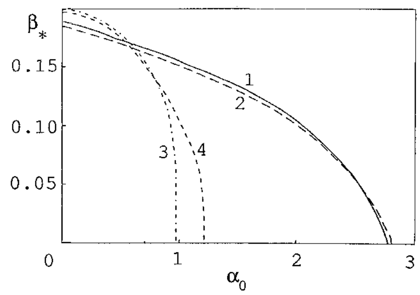

Fig. 5 Critical value β_* as a function of the parameter α_0 (1 – γ = 5, æ = 0.001, L = 0.5; 2 – γ = 5, æ = L = 0; 3 – γ = 2.5, æ = 0.001, L = 0.5; and 4 – γ = 2.5, æ = L = 0).

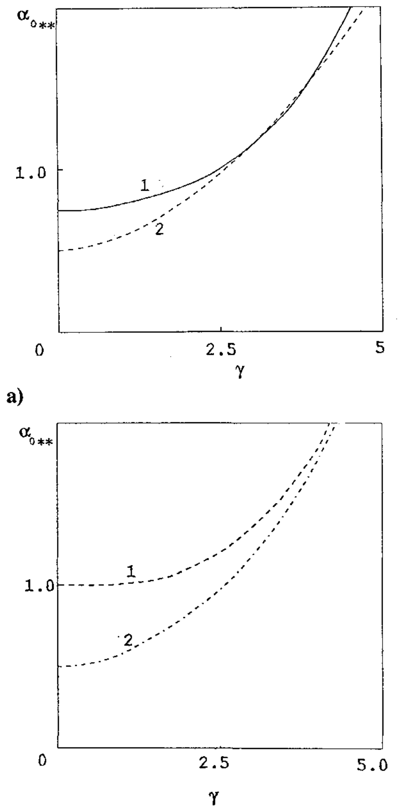

Fig. 6 Critical value α_{0**} as a function of the parameter γ (1 – æ = L = 0; 2 – æ = 0.001, L = 0.5).

Curves, which are shown in Figs. 2–4, illustrate the influence of the parameter ε (of the external and aerodynamic damping) (Fig. 2), the parameter æ, characterizing the relaxation time of the plate material (Fig. 3), and the parameter L, characterizing the relaxation measure (Fig. 4) on the value of the critical parameter β_* at $\gamma = 5$. The curve of the relationship $\beta_* \sim \varepsilon$ for the elastic plate (æ $= L = 0$) in Fig. 2 practically coincides with the curve for the viscoelastic plate at $lg\,æ = -6$. These results testify that the

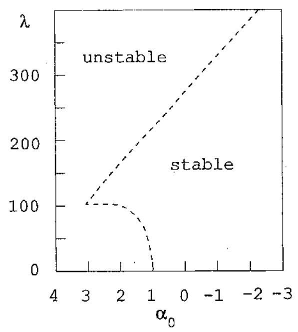

Fig. 7 Mean square stability boundaries for the value of random spectral density of the load D_x = 5.

value of the parameter β_* is decreased with the decrease of the parameters ε, æ, and L. Let us notice that the similar situation is also valid for the viscoelastic rod loaded by conservative ("dead") force.[17,18] It is interesting to notice that the plate may be stable in mean square at $L = 1$ (Fig. 4). The equality $L = 1$ means that the plate material is Maxwell material, which has unlimited creep. The situation in the conservative problem was another,[17,18] i.e., the rod was unstable in mean and in mean square at $L = 1$. Calculations show that the value of the critical parameter β_* for the viscoelastic plate is equal to the value β_* for the elastic plate, if parameters æ and L are nearing zero, i.e.,

$$\lim_{æ \to 0} \beta_*(\varepsilon, æ) = \beta_*(\varepsilon, 0)$$

$$\lim_{L \to 0} \beta_*(\varepsilon, L) = \beta_*(\varepsilon, 0)$$

If all parameters of damping ε, æ, and L, tend to zero, then the value β_* tends to zero too.

Curves of the relationship between parameters β_* and α_0 for elastic and viscoelastic plates at $\varepsilon = 0.01$ are presented in Fig. 5. It can be noticed that $\beta_* = 0$ on this curve corresponds to $\alpha_0 = \alpha_{0*}$, which is the critical value of parameter α_0 by the consideration of the stability in mean (or by the deterministic treatment of the problem). One can see that the value of the critical parameter β_* begins to decrease sharply at $\alpha_0 \to \alpha_{0*}$. The velocity of this decrease increases with the decrease of the parameter γ (of the flow speed).

The charts of the critical parameter α_{0**} (stability in mean square) from the parameter γ for the elastic and viscoelastic plates at the fixed value of the parameter $\beta = 0.1$ and at $\varepsilon = 0.01$ are shown at Fig. 6a. For the comparison, the similar charts in the case of the stability in mean ($\beta = 0$) are shown at Fig. 6b.

Let us notice that the last item in Eq. (1) takes into account the interaction of the flow and the plate not quite right at small values of the parameter γ (at small speeds of the flow). Therefore the solution obtained at small values of the parameter γ corresponds to the validity not quite exactly.

The solution of the problem for the plate at $\gamma = 0$ coincides with the solution for the plate subjected only to the random load. The problem in such case is conservative and the corresponding solution can be found from the solution for the viscoelastic rod.[17,18]

The chart in Fig. 7 is similar to chart 1 in Fig. 6a. This curve agrees with the identical curve from Ref. 12. For the convenience of the comparison, other notations are used, which coincide with notations from Ref. 12. $\lambda = \pi^4\gamma/4$, $\alpha_0 = -R_x^0/\pi^2$, $\varepsilon = 0.1\lambda^{\frac{1}{2}}/(2\pi^2)$, and $D_x = \pi^8\beta^2/2$.

Conclusions

The stability of the viscoelastic plate, subjected to a random stationary load in a supersonic flow, is investigated. Random fluctuations of the load are proportional to Gaussian white noise. The relaxation kernel of the material is assumed in the form of the sum of exponents. The influence of the external damping, viscous properties of the material, the flow speed, and probabilistic parameters of the load on the asymptotical stability in mean square is analyzed. It is underlined that the viscoelastic plate may be asymptotically stable in mean square in the case, if the plate material is Maxwell's material, i.e., the material with the unlimited creep.

It should be noted that the approximate solution of the problem, based on the presentation of the transverse deflection of the plate as the sum of the two sines, was obtained. The principal problem of the present article is the qualitative investigation of the stability in mean square of the plate. Of course, the more exact result can be found, if we keep the number of the terms in the expression (2) more two. This yields, however, the sharp increase of the number of equations of the system (12) and complicates significantly the analysis of roots of the corresponding characteristical equation.

Previously, the case was considered, when only the exterior load is a stochastic process. In reality, the case can present the interest, when, besides the load, other factors are random processes too, for example, the flow speed and characteristics of the material viscosity and of the exterior damping. If random fluctuations of these processes are presented as noncorrelative white noises, then, for the solution of the problem, the method can be used, which was proposed by the author[20] for the viscoelastic rod, subjected to the conservative random force.

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